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Long-range connections and mixed diffusion in fractional networks

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Abstract

Networks with long-range connections obeying a distance-dependent power law of sufficiently small exponent display superdiffusion, Lévy flights and robustness properties very different from the scale-free networks. It has been proposed that these networks, found both in society and biology, be classified as a new structure, *the fractional networks*. Particular important examples are the social networks and the modular hierarchical brain networks where both short- and long-range connections are present. The anomalous superdiffusive and the mixed diffusion behavior of these networks is studied here as well as its relation to the nature and density of the long-range connections.

1 Introduction

The human brain contains up to 86 billion neurons connected by close to a million kilometers of axons and dendrites. Most of these connections ($\sim$80%) are short range on the order of a few hundred microns, the rest ($\sim$20%) being long-range myelinated fibers on the order of several centimeters. The insulating myelin sheath increases conduction velocity of the action potentials but at the cost of taking up more volume in the brain as well as rendering axons unable to synapse onto nearby neurons. That evolution has found profitable to accept this additional hardware cost highlights the importance of long-range connections.

From a network point of view the brain has a modular and hierarchical structure [1] [2]. Each module is associated to a specialized function mediated by short-range connections whereas global integration, for higher cognition functions, relies on the long-range connections between modules.

The existence and importance of long-range connections in the brain has been much studied in recent years [3] [4] [5] [6] [7] [8], with diminished long-range

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functional connectivity being associated to cognitive disorders [9]. Of course, by itself, existence of long-range connections between the specialized nodes does not guarantee global integration of the cognitive functions. It is also necessary that the flow of information be sufficiently fast for the stimulus integration to be performed in a timely manner. This seems of particular relevance for the forward and backwards loops in the predictive coding mode [10] [11] [12] [13] [14] [15] of brain operation. One may therefore ask what type of communication short- and long-range connections establish and whether it depend or not on the structure and density of the long-range connections.

The network modules in the brain are in fact repertoires of many neurons and, when dealing with the interactions of these intrinsic connectivity networks (ICN's), a continuous diffusion approximation might be a good modelling hypothesis. In another paper [16] the nature of the diffusion processes associated to short and long-range connections have been analyzed. In particular it was concluded that whereas for short-range connections information propagates as a normal diffusion, for long-range connections of a certain type, one has anomalous diffusion, sub- or super-diffusion depending on the power law distance dependence of the connections.

Networks with long-range connections leading to superdiffusion display properties so very different from scale-free and hub dominated networks that in [16] it was proposed to characterize them as a new class of networks, **the fractional networks**. Notice that long-range connections are also important in social networks [17] [18] [19] [20].

In a network the Laplacian matrix is

$$L = G - A \tag{1}$$

G being the degree matrix ($G_{ij} = \delta_{ij} \times$ number of connections of node i) and A the adjacency matrix ($A_{ij} = 1$ if i and j are connected, $A_{ij} = 0$ otherwise). Let $\psi(i)$ for each node i be the intensity of some function ψ across the network. For a node i connected along some coordinate to two other nearest neighbor nodes $i + 1$ and $i - 1$ the action of the Laplacian matrix on a vector leads to $-\psi(i-1) + 2\psi(i) - \psi(i-1)$, which is a discrete version of $-d^2$ (minus the second derivative). It is reasonable to think that ψ diffuses from i to j proportional to $\psi(i) - \psi(j)$ if i and j are connected. Then,

$$\frac{d\psi(i)}{dt} = -k \sum_j A_{ij} (\psi(i) - \psi(j)) = -k \left(\psi(i) \sum_j A_{ij} - \sum_j A_{ij}\psi(j) \right) \tag{2}$$

which in matrix form is

$$\frac{d\psi}{dt} + kL\psi = 0 \tag{3}$$

a heat-like equation. Therefore the Laplacian matrix controls the diffusion of quantities in the network and in the continuous approximation and for short-range connections the propagation of signals in the network may be represented

by a normal diffusion equation.

$$\frac{d\psi}{dt} = k\Delta\psi \tag{4}$$

Δ being the Laplacian in the dimension of the space where the network is embedded.

However, for long-range connections the situation is different and from the symmetrized Grünwald-Letnikov representation of the fractional derivative it was found [16] (see also the Appendix) that for networks where the probability of establishment of a link at distance d is proportional to a power of the distance

$$P_{ij} = cd_{ij}^{-\gamma} \tag{5}$$

diffusion would be fractional diffusion of exponent $\beta = \gamma - 1$. $\beta = 2$ being normal diffusion and all $\beta < 2$ corresponding to superdiffusions.

$$\frac{d\psi}{dt} = -k\left(-\Delta\right)^{\frac{\beta}{2}}\psi \tag{6}$$

Anomalous diffusion and other phenomena [16] emerge naturally as a structural property in long-range connection networks with distance dependence as in (5). Here, in Section 2, the case of networks characterized by a modular hierarchical structure with both short and long range connections will be studied. This is the structure that occurs in brain networks and also in some social networks. Whereas in the networks studied in [16] the uniform scaling law of the connections leads to pure anomalous diffusion, here one faces a mixture of both normal and anomalous diffusion. This is the central phenomena that is studied in this paper with emphasis on the nature of the time scales of propagation of information. This is discussed in the framework of the continuous approximation to the network leading to a fractional differential equation. The continuous approximation is a reasonable approximation for very large networks. However it is also found that qualitatively similar results are obtained even for small discrete networks. This is illustrated in Section 3, by numerical simulation in a relatively small network (400 nodes).

2   Mixed diffusion

In the mixed case the diffusion equation will be

$$\frac{d\psi\left(x,t\right)}{dt} = \left(a\Delta - b\left(-\Delta\right)^{\frac{\beta}{2}}\right)\psi\left(x,t\right) \tag{7}$$

with $x \in \mathbb{R}^n$, n being the dimension of the embedding Euclidean space. For the Fourier transform

$$\widetilde{\psi}\left(k,t\right) = \int d^n x \psi\left(x,t\right) e^{-ik\cdot x} \tag{8}$$
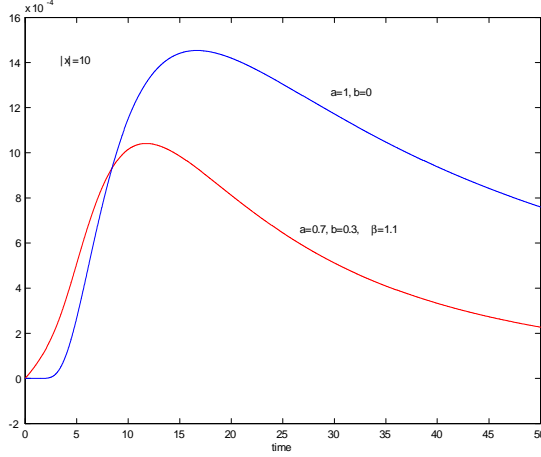
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Figure 1: Comparison of the propagation time of a delta signal at $(x = 0, t = 0)$ to a distance $x = 10$ for normal and mixed diffusion ($\beta = 1.1, a = 0.7, b = 0.3$)

one has the equation

$$\frac{d\widetilde{\psi}(k,t)}{dt} = \left(-a\,|k|^2 - b\,|k|^\beta\right)\widetilde{\psi}(k,t) \tag{9}$$

with solution

$$\widetilde{\psi}(k,t) = \widetilde{\psi}(k,0)\, e^{-t\left(a|k|^2 + b|k|^\beta\right)} \tag{10}$$

$\widetilde{\psi}(k,0) = 1$ corresponds to $\psi(x,0) = \delta^{(n)}(x)$, that is, an initial localized disturbance at the origin. This is the situation of interest to study the propagation of information in the network. Computing the inverse Fourier transform one has

$$\psi(x,t) = \frac{2A_n}{(2\pi)^{n-1}} \int_0^\infty d\,|k|\,|k|^{n-1}\, e^{-t\left(a|k|^2 + b|k|^\beta\right)} \frac{\sin(|k|\,|x|)}{|k|\,|x|} \tag{11}$$

with

$$A_n = \begin{cases} \frac{\pi^{m-1} 2^{m-2}}{(2m-2)!!} & n = 2m \\ \frac{\pi^{m-1} 2^{m-1}}{(2m-1)!!} & n = 2m+1 \end{cases} \tag{12}$$

As in the purely fractional multidimensional solution [21] one notices the strong dependence on the dimension n.

Numerical evaluation of (11) shows the remarkable difference in the speed of propagation of information between normal and mixed diffusion. For $n = 3$, Figures 1 and 2 compare the propagation of a delta signal at $(x = 0, t = 0)$ to distances $x = 10$ and 100 for normal and mixed diffusion. One sees that whereas for normal diffusion it takes a long time for the signal to be detected at a distance, for mixed diffusion the behavior is qualitatively very different.

Figures 3 and 4 show that this effect is obtained even with a very small amount of fractional diffusion.
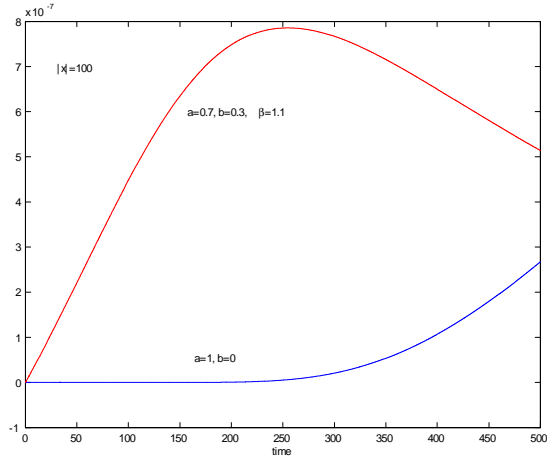
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Figure 2: Comparison of the propagation time of a delta signal at $(x = 0, t = 0)$ to a distance $x = 100$ for normal and mixed diffusion ($\beta = 1.1, a = 0.7, b = 0.3$)
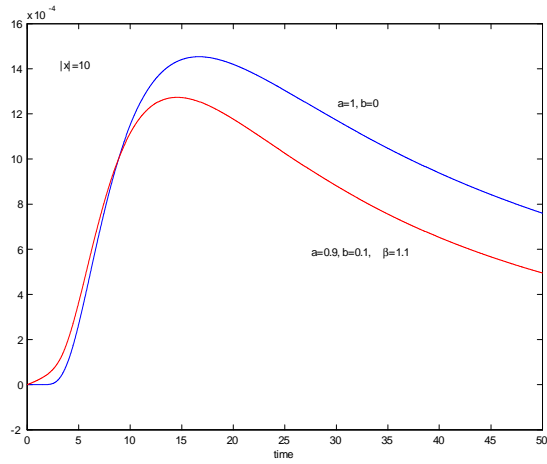

Figure 3: Comparison of the propagation time of a delta signal at $(x = 0, t = 0)$ to a distance $x = 10$ for normal and mixed diffusion ($\beta = 1.1, a = 0.9, b = 0.1$)
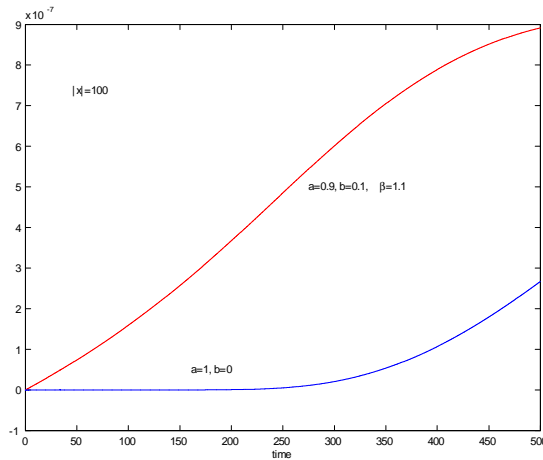
Figure 4: Comparison of the propagation time of a delta signal at $(x = 0, t = 0)$ to a distance $x = 100$ for normal and mixed diffusion ($\beta = 1.1, a = 0.9, b = 0.1$)

Of course the effect exist only if $\beta < 2$. For $\beta \geq 2$ the behavior would be practically indistinguishable from normal diffusion. This puts into evidence the fact that the mere existence of long-range connections does not guarantee the existence of fractional superdiffusion. That is, a sufficient density of long-range connections to be at least consistent with the one in (5) is required. This is an important hint to be taken into account on the relation of functional connectivity to brain cognitive disorders.

3 Diffusion in a small fractional network: Numerical results

So far we have discussed the diffusion behavior of fractional networks in the framework of the continuous approximation to the network. Here, by numerically simulating the propagation of a pulse of information in a discrete network, we show that the results are consistent with those obtained from the continuous approximation modeled by the fractional differential equations.

We consider 400 agents (nodes) placed in two-dimensional 20x20 grid and establish connections among the nodes with a distance-dependent power-law distribution

$$P_{ij} \tilde{\ } d^{-\gamma}$$

Namely, we pick a node at random and establish a connection to another node at a distance d

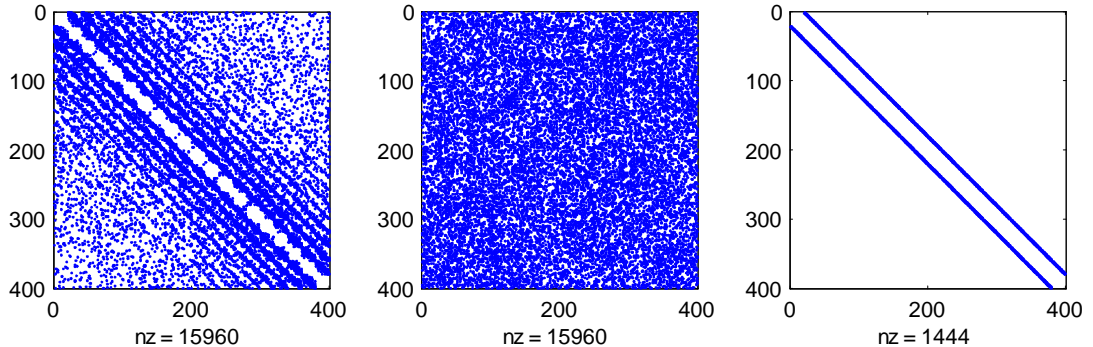$$d = \exp \left\{ \frac{\log \left(d_{\min}^{1-\gamma} - C\gamma y \right)}{1 - \gamma} \right\}$$

6

Figure 5: Connection patterns for a fractional network with $\gamma = 2$ and sparsity 0.1, for a random network with the same sparsity and a neares-neighbor network.

y being a random number in the interval $[0, 1]$ and C a constant

$$C = \frac{\left(d_{\min}^{1-\gamma} - d_{\max}^{1-\gamma}\right)}{\gamma}$$

In Fig.5 we show the pattern of connections for the fractional network with $\gamma = 2$ and a sparsity index of 0.1. In the same figure are also shown the patterns of connections for a random network with the same sparsity and for a nearest-neighbor network with the maximal number of connections.

To study the diffusion in the fractional network, we consider, at time zero, a unit pulse at one node and study how it propagates throughout the network. At each time step the pulse is transmitted to the neighbors of each activated node, with a no-backflow condition being imposed. That is, after a node transmits the pulse to its neighbors it no longer transmits the same pulse even if it receives it back through some cycle in the network. In the Fig.6 we show the results of two typical simulations. In each case we have chosen, among the nodes that have a long-distance connection, those that are further apart. In the same figures we compare with the results of the same experiment for a nearest-neighbor network (the single pulses at times 16 and 20). Not only is the signal transmitted much faster in the fractional network, but also its coherent nature is preserved, instead of being spread over a very large number of distinct times as it may occur in a sparse random network. Very similar behavior is also obtained for the propagation between nodes that are not directly connected at time one.

4   Remarks and conclusions

1. As has been experimentally confirmed, existence of long-range connections between the brain ICN's is critical for integration and interpretation of sensory stimuli and higher cognitive functions. One view of brain integration and consciousness [22] [23] is based on a percolation model. For percolation, that is, for
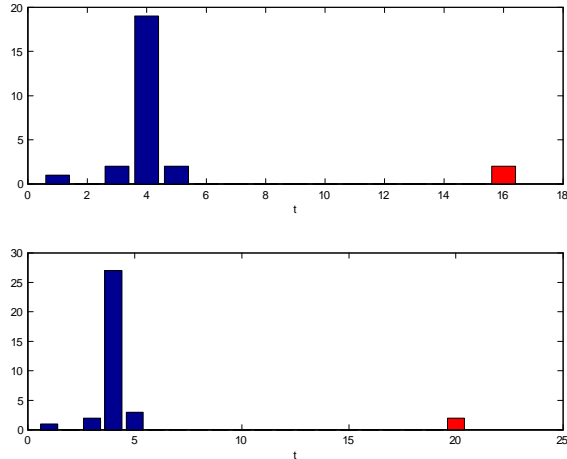
Figure 6: Propagation of a unit pulse between two distant nodes for a fractional network ($\gamma = 2$), compared with the same phenomenon in a nearest-neighbor network.

the formation of a global cluster, it suffices that connections exist between the local clusters. However for the establishment of higher cognitive functions, and in particular in the predictive coding mode, it is necessary that the interaction between the ICN's be established at a sufficiently fast rate. Therefore the mere existence of long-range connections is not sufficient, it also necessary that they have, for example, a power-law dependence with $\gamma < 3$.

2. The additional hardware cost of myelinated long-range connections in the brain is compensated by the integration of information and higher cognitive functions. Another puzzling additional energetic cost is that, when tested with fMRI, the resting brain is in fact turbulent and restless [24]. There is a good reason for that, probably related to speed of reaction. With the operating time scales of individual neurons and their low average firing rate, pattern recognition by evolution towards an equilibrium fixed point or minimizing an energy function would be much too slow for practical living purposes. As has been conjectured, for example from the studies of the olfactory bulb [25], a much faster recognition is achieved by replacing the low-level chaos that exists in the absence of an external stimulus by, in the presence of a signal, a pattern of bursts with different intensities in different regions. A network of Bernoulli units [26] is a model confirmation of this conjecture.

3. Finally, as already discussed in [16], the robustness and controllability properties of the fractional networks are so very different from the scale-free networks that they deserve a detailed study. This is relevant not only for brain functions but also concern the uses and misuses of information flow in the social networks.

5   Appendix: Power-law long-range connections and fractional diffusion

For completeness we include here a short derivation of the relation between power-law long-range connections and fractional diffusion equations, already discussed in Ref.[16].

Let the probability of a link at distance d be proportional to a power of the distance

$$P_{ij} = c d_{ij}^{-\gamma} \qquad\qquad \text{with } \gamma \leq 3$$

Consider now a block renormalized network N^* where each set of q nearby nodes in the original network N are mapped to a node of the N^* network. With the block renormalization, the power-law connection probability leads to actual connection strengths in the renormalized network. In the N^* network the connections are

$$A_{ij}^* \simeq c q d_{ij}^{-\gamma}$$

with the Laplacian L^* and degree G^* matrices of the N^* network being

$$L^* \psi(i) = G_{ii}^* \psi(i) - cq \sum_{j \neq i} d_{ij}^{-\gamma} \psi(j)$$

Compare the distance dependence of the elements of the Laplacian matrix L^* along one of the coordinate axis with a discrete one-dimensional representation of a fractional derivative. The symmetrized Grünwald-Letnikov representation of the fractional derivative $(a < x < b)$ (see [27]) is

$$
\begin{aligned}
D^\beta \psi(x) \;=\; & \frac{1}{2} \lim_{h \to 0} \frac{1}{h} \left\{ \sum_{n=0}^{\left[\frac{x-a}{h}\right]} (-1)^n \begin{pmatrix} \beta \\ n \end{pmatrix} \psi(x - nh) \right. \\
& \left. + \sum_{n=0}^{\left[\frac{b-x}{h}\right]} (-1)^n \begin{pmatrix} \beta \\ n \end{pmatrix} \psi(x + nh) \right\}
\end{aligned}
\tag{13}
$$

with coefficients

$$
\left| \begin{pmatrix} \beta \\ n \end{pmatrix} \right| = \frac{\Gamma(\beta+1) \left|\sin(\pi\beta)\right|}{\pi} \frac{\Gamma(n-\beta)}{\Gamma(n+1)} \underset{n \text{ large}}{\frown} \frac{\Gamma(\beta+1) \left|\sin(\pi\beta)\right|}{\pi} n^{-(\beta+1)}
\tag{14}
$$

and $sign \begin{pmatrix} \beta \\ n \end{pmatrix} = (-1)^{n+1}$.

Comparing (13-14) with the expression for $L^* \psi(i)$, the conclusion is that diffusion in the N^* network is fractional diffusion of exponent $\beta = \gamma - 1$. $\beta = 2$ would be normal diffusion, all $\beta < 2$ corresponding to superdiffusions.

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